Stationarity

Definition, meaning and consequences

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November 14, 2008

Version 1.1

Outline

Economies of scale and increasing returns

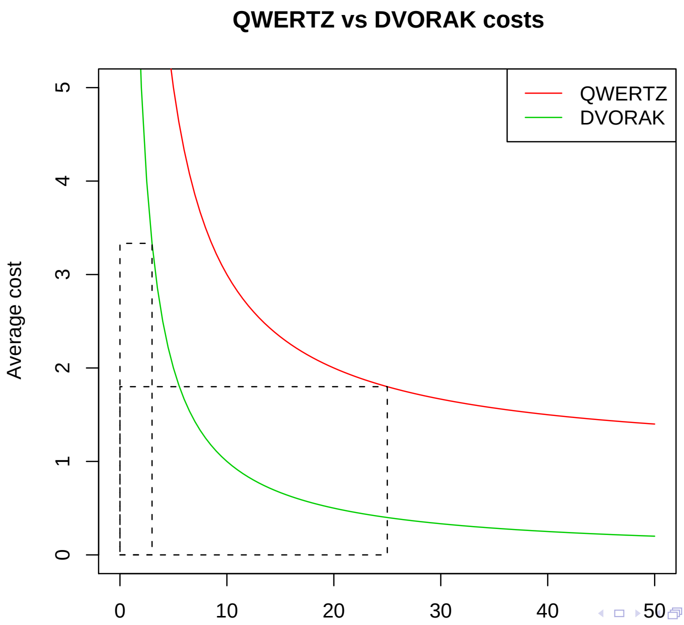
So-called Nobel Prize has been attributed to Krugman:

> *Economies of scale combined with reduced transport costs also help to explain why an increasingly larger share of the world population lives in cities and why similar economic activities are concentrated in the same locations. [This], in turn, stimulates further migration to cities.*

The increasing returns are:

- Self production: Learning curve
- Common production: adoption/network externalities

How is a keyboard organised?



Outline

1. Stationarity
2. ARMA models for stationary variables
3. Some extensions of the ARMA model
4. Non-stationarity
5. Seasonality
6. Non-linearities
7. Multivariate models
8. Structural VAR models
9. Cointegration the Engle and Granger approach
10. Cointegration 2: The Johansen Methodology
11. Multivariate Nonlinearities in VAR models
12. Multivariate Nonlinearities in VECM models

Outline

Outline

What is a time serie?

In econometrics, we deal with three types of data:

- Cross individual data: $X_i \quad i$: individu
- Times series data: $X_t \quad t$: time
- Panel data: $X_{it} \quad t$: time

Definition

A time series is a variable X indexed by the time t: $X_t \qquad t = 1, 2, \ldots, T$.

Examples

The time t can be annual, monthly, daily...

- Let be X_t the annual GDP.
- Let Y_t be the monthly temperature
- Z_t be the daily stocks

Some distinctions

The time series can:

- Take discrete or continuous values.
- Be measured at discrete (monthly, daily) or continous time (signal processing, finance).
- Be measured at regular or irregular intervals.

In this lecture, usually we will refer to time series that take continous values and are measured at discrete and regular intervalls.

Outline

Description of a time serie

To describe a time serie, we will concentrate on:

- Its Data Generating Process (DGP)
- The joint distribution of its elements
- Its "moments":
    - Its expected value, $\mathrm{E}[X_t] \equiv \mu_t$.
    - Its variance, $\mathrm{Var}[X_t] \equiv \sigma^2 \equiv \gamma_0$.
    - Its autocovariance or autocorrelation of order k, $\mathrm{Cov}[X_t, X_{t-k}] \equiv \gamma_k$.

Variance

The "second" moments play an important role in time series and have a slight different definition:

Definition

The **variance** of X_t, denoted by $\gamma_0(t)$, is given by:
$\mathrm{Var}(X_t) \equiv \mathrm{E}\left[(X_t - \mathrm{E}(X_t))^2\right]$

Definition

The **autocovariance** of X_t of order k, denoted by $\gamma_k(t)$, is given by:
$\mathrm{Cov}(X_t, X_{t-k}) \equiv \mathrm{E}\left[(X_t - \mathrm{E}(X_t))(X_{t-k} - \mathrm{E}(X_{t-k}))\right]$

Definition

The **autocorrelation** of X_t of order k, denoted by $\rho_k(t)$, is given by:
$\mathrm{Corr}(X_t, X_{t-k}) \equiv \frac{\mathrm{Cov}(X_t, X_{t-k})}{\sqrt{\mathrm{Var}(X_t)}\sqrt{\mathrm{Var}(X_{t-k})}}$

Outline

The stationarity is an essential property to define a time series process:

Definition

A process is said to be **covariance-stationary**, or **weakly stationary**, if its first and second moments are time invariant.

$$
\begin{array}{ll}
E(Y_t) = E[Y_{t-1}] = \mu & \forall\, t \\
\mathrm{Var}(Y_t) = \gamma_0 < \infty & \forall\, t \\
\mathrm{Cov}(Y_t, Y_{t-k}) = \gamma_k & \forall\, t, \forall\, k
\end{array}
$$

Weak stationarity

- The third condition states that the autocovariances only depend on the decay in the time but not in the time itself.
- Hence, the structure of the serie does not change with the time.
- If a process is stationary, $\gamma_k = \gamma_{-k}$

Mean reverting propriety

A stationary process has the propriety to be *mean reverting*:

- it will fluctuate around its mean.
- This mean will act as an attractor.
- It will cross the mean line an infinite number of ways.

Strong stationarity

Definition

$f(Z_1, Z_2, ..., Z_t) = f(Z_{1+k}, Z_{2+k}, ..., Z_{t+k})$

.

- Implies weak stationarity
- Definition not very useful as informations about the density function are difficult to obtain

ergodicity

Not absolutely useful...

Definition

$\lim_{n\to\infty} \mathrm{E}\left[|Y(y_i,\ldots,y_{i+k})Z(y_{i+n},\ldots,y_{i+n+l})|\right] =$
$\mathrm{E}\left[Y(y_i,\ldots,y_{i+k})\right]\cdot\mathrm{E}\left[Z(y_{i+n},\ldots,y_{i+n+l})\right]$

Outline

Example 1

We define now the simplest stationary process:

Definition

An **independent white noise process** is a sequence of independant elements with same expected value and variance.

We will denote it $\varepsilon_t \sim iid(0, \sigma^2)$

Example 2

Consider the first order auto-regressive process AR(1) with a constant:

$$Y_t = c + \varphi Y_{t-1} + \varepsilon_t \qquad \varepsilon_t \sim iid(0, \sigma^2)$$

Theorem

An AR(1) process is asymptotically stationary if $|\varphi| < 1$

We will need fot its proof to remember the properties of a geometric progression:

Lemma (Infinite geometric progression)

$|\alpha| < 1 \Leftrightarrow \sum_{i=0}^{\infty} \alpha^i = 1 + \alpha + \alpha^2 + \alpha^3 + \ldots = \frac{1}{1-\alpha}$

Proof.

The AR(1) can be written as:

$$Y_t = c\sum_{i=0}^{t-1}\varphi^i + \varphi^t Y_0 + \sum_{i=0}^{t-1}\varphi^i\varepsilon_{t-i}$$

If $|\varphi| < 1$, it can be simplified into:

$$Y_t = \frac{c}{1-\varphi} + \sum_{i=0}^{t-1}\varphi^i\varepsilon_{t-i}$$

We have then:

- $\mathrm{E}(X_t) = \frac{c}{1-\varphi}$
- $\mathrm{Var}(X_t) = \frac{\sigma^2}{1-\varphi^2}$
- $\mathrm{Cov}(X_t, X_{t-j}) = \frac{\varphi^j}{1-\varphi^2}\sigma^2$

□

Outline

Corollary

An AR(1) process is nonstationary if $|\varphi| \geq 1$. *Furthemore, if:*

- $|\varphi| = 1$ *it is difference stationary*
- $|\varphi| > 1$ *it is explosive.*

Outline

Proof.

The AR(1) process without constant $Y_t = \varphi Y_{t-1} + \varepsilon_t \qquad \varepsilon_t \sim iid(0, \sigma^2)$ with $|\varphi| = 1$ can be rewritten as:

$$Y_t = Y_0 + \sum_{i=0}^{t-1} \varepsilon_{t-i}$$

We have then:

- $\mathrm{E}(X_t) = Y_0$
- $\mathrm{Var}(X_t) = t\sigma^2$
- $\mathrm{Cov}(X_t, X_{t-j}) = (t-j)\sigma^2$

□

The AR(1) process with $|\varphi| = 1$ is called a random walk. It is said to be difference stationary.

Definition

The difference operator takes the difference between a value of a time serie and its lagged value. $\Delta X_t \equiv X_t - X_{t-1}$

Definition

A process is said to be difference stationary if it becomes stationary after being differenced once.

Note: a difference stationary process is also called integrated of order 1 and denoted by $X_t \sim I(1)$

Theorem

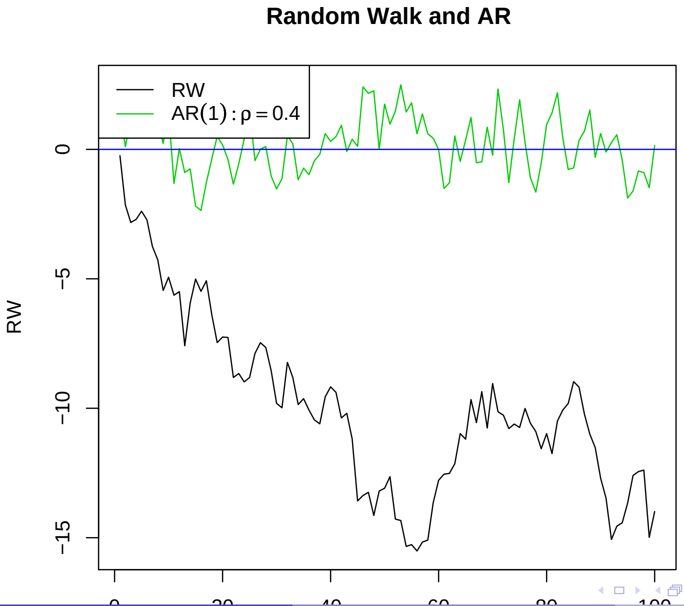
A random walk is difference stationary.

Proof.

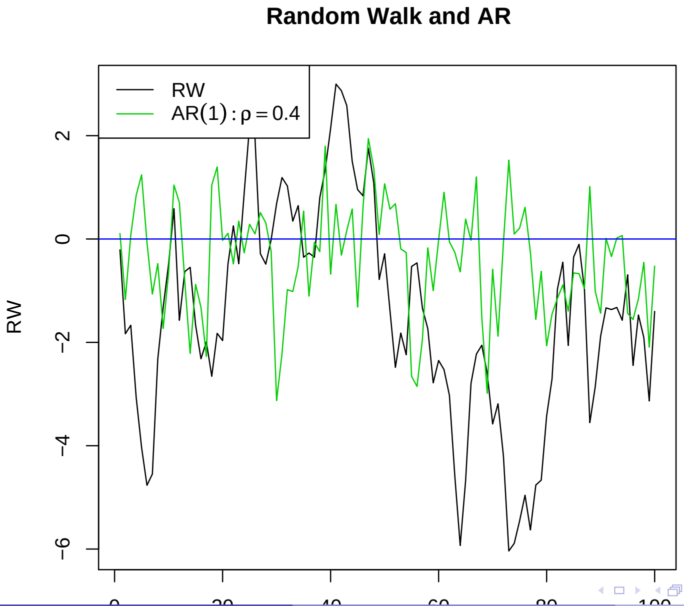
$$\Delta Y_t = Y_t - Y_{t-1} = Y_{t-1} + \varepsilon_t - Y_{t-1} = \varepsilon_t \qquad \square$$

Stationary AR(1)	Random Walk
$Y_t = \varphi Y_{t-1} + \varepsilon_t$	$Y_t = Y_{t-1} + \varepsilon_t$
$\vert\varphi\vert < 1, \varepsilon_t \sim iid(0, \sigma^2)$	$\varepsilon_t \sim iid(0, \sigma^2)$
$Y_t = Y_0 + \sum_{i=0}^{t-1} \varphi^i \varepsilon_{t-1}$	$Y_t = Y_0 + \sum_{i=0}^{t-1} \varepsilon_{t-1}$
$\mathrm{E}(Y_t) = 0(\text{ if } Y_0 = 0)$	$\mathrm{E}(Y_t) = 0(\text{ if } Y_0 = 0)$
$\mathrm{Var}(Y_t) = \frac{\sigma^2}{1-\varphi^2}$	$\mathrm{Var}(Y_t) = \sigma^2 t$
$\mathrm{Cov}(Y_t, Y_{t-j}) = \frac{\varphi^j}{1-\varphi^2}\sigma^2$	$\mathrm{Cov}(Y_t, Y_{t-j}) = (t-j)\sigma^2$
$\mathrm{Corr}(Y_t, Y_{t-j}) = \varphi^j$	$\mathrm{Corr}(Y_t, Y_{t-j}) = \sqrt{\frac{t-s}{t}} \xrightarrow{\infty} 1$
$\frac{\delta Y_t}{\delta \varepsilon_{t-i}} \xrightarrow{\infty} 0$	$\frac{\delta Y_t}{\delta \varepsilon_{t-i}} = 1$
Is mean reverting	Tends to move away from the mean

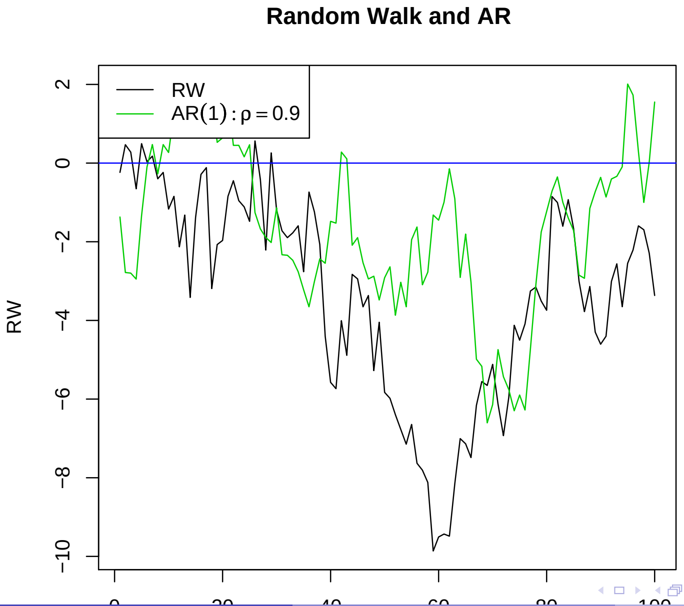
Simulation of AR(1) and RW



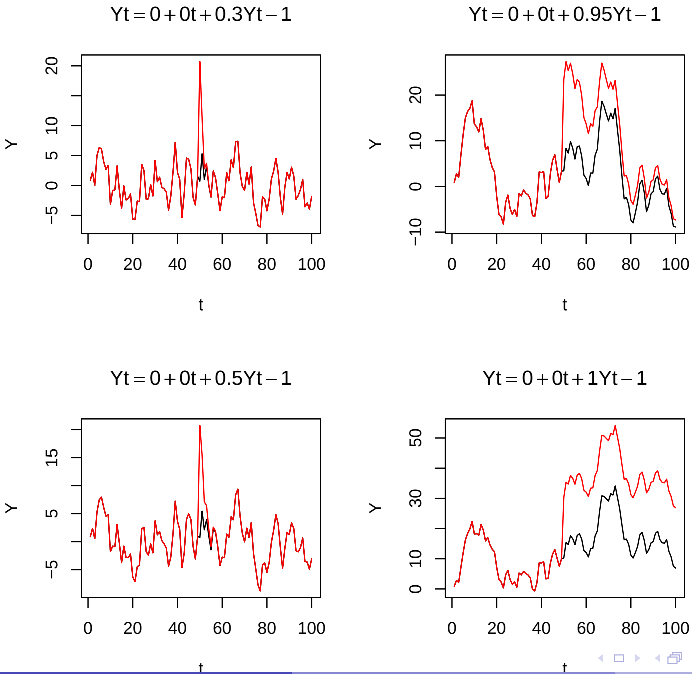
Simulation of AR(1) and RW



Simulation of AR(1) and RW



Stationary AR(1)	Random Walk
$Y_t = \varphi Y_{t-1} + \varepsilon_t$	$Y_t = Y_{t-1} + \varepsilon_t$
$\lvert\varphi\rvert < 1, \varepsilon_t \sim iid(0, \sigma^2)$	$\varepsilon_t \sim iid(0, \sigma^2)$
$Y_t = Y_0 + \sum_{i=0}^{t-1} \varphi^i \varepsilon_{t-1}$	$Y_t = Y_0 + \sum_{i=0}^{t-1} \varepsilon_{t-1}$
$\mathrm{E}(Y_t) = 0(\text{ if } Y_0 = 0)$	$\mathrm{E}(Y_t) = 0(\text{ if } Y_0 = 0)$
$\mathrm{Var}(Y_t) = \frac{\sigma^2}{1-\varphi^2}$	$\mathrm{Var}(Y_t) = \sigma^2 t$
$\mathrm{Cov}(Y_t, Y_{t-j}) = \frac{\varphi^j}{1-\varphi^2}\sigma^2$	$\mathrm{Cov}(Y_t, Y_{t-j}) = (t-j)\sigma^2$
$\mathrm{Corr}(Y_t, Y_{t-j}) = \varphi^j$	$\mathrm{Corr}(Y_t, Y_{t-j}) = \sqrt{\frac{t-s}{t}} \xrightarrow{\infty} 1$
$\frac{\delta Y_t}{\delta \varepsilon_{t-i}} \xrightarrow{\infty} 0$	$\frac{\delta Y_t}{\delta \varepsilon_{t-i}} = 1$
Is mean reverting	Tends to move away from the mean



Yt = 0 + 0t + 0.3Yt − 1
Yt = 0 + 0t + 0.95Yt − 1
Yt = 0 + 0t + 0.5Yt − 1
Yt = 0 + 0t + 1Yt − 1
Y
t

Outline

The random walk with drift

$X_t = a + x_{t-1} + \varepsilon_t \qquad \varepsilon_t \sim iid(0, \sigma^2)$
Can be rewritten as:

$$Y_t = at + Y_0 + \sum_{i=0}^{t-1} \varepsilon_{t-i}$$

Expectation is also time-varying:

- $\mathrm{E}(X_t) = Y_0 + at$
- $\mathrm{Var}(X_t) = \ldots = f(t)$
- $\mathrm{Cov}(X_t, X_{t-j}) = \ldots = f(t)$

But it is still difference stationary:

Proof.

$\Delta Y_t = Y_t - Y_{t-1} = a + Y_{t-1} + \varepsilon_t - Y_{t-1} = a + \varepsilon_t$ $\square$

Outline

Trend stationarity

$$Y_t = a + bt + \varphi Y_{t-1} + \varepsilon_{t-1}$$

$$Y_t = a \sum_{i=0}^{t-1} \varphi^i + a \sum_{i=0}^{t-1} \varphi^i + \varphi^t Y_0 + \sum_{i=0}^{t-1} \varphi^i \varepsilon_{t-i}$$

If $|\varphi| < 1$, it can be simplified into:

$$Y_t = \frac{a}{1-\varphi} + b \sum_{i=0}^{t-1} \varphi^i (t-i) + \sum_{i=0}^{t-1} \varphi^i \varepsilon_{t-i}$$

Proof.

$$
\begin{aligned}
Y_t &= a + bt + \varphi Y_{t-1} + \varepsilon_{t-1} \\
&= a + bt + \varphi(a + b(t-1) + \varphi Y_{t-2} + \varepsilon_{t-2}) + \varepsilon_{t-i} \\
&= a(1+\varphi) + bt + \varphi b(t-1) + \varphi^2 Y_{t-2} + \varphi\varepsilon_{t-2} + \varepsilon_{t-1} \\
&= a(1+\varphi) + bt + \varphi b(t-1) + \varphi^2(a + b(t-2) + \varphi Y_{t-3} + \varepsilon_{t-3}) \\
&\quad + \varphi\varepsilon_{t-2} + \varepsilon_{t-1} \\
&= a(1+\varphi+\varphi^2) + b(t + \varphi(t-1) + \varphi^2(t-2)) + \varphi^3 Y_{t-3} \\
&\quad + \varphi\varepsilon_{t-3} + \varphi\varepsilon_{t-2} + \varepsilon_{t-1} \\
&= \dots \\
&= \varphi^t Y_0 + a\sum_{i=0}^{t-1}\varphi^i + b\sum_{i=0}^{t-1}\varphi^i(t-i) + \sum_{i=0}^{t-1}\varphi^i\varepsilon_{t-i} \\
&= \frac{a}{1-\varphi} + b\sum_{i=0}^{t-1}\varphi^i(t-i) + \sum_{i=0}^{t-1}\varphi^i\varepsilon_{t-i}
\end{aligned}
$$

Thus the model $Y_t = a + bt + \varphi Y_{t-1} + \varepsilon_{t-1}$ has:

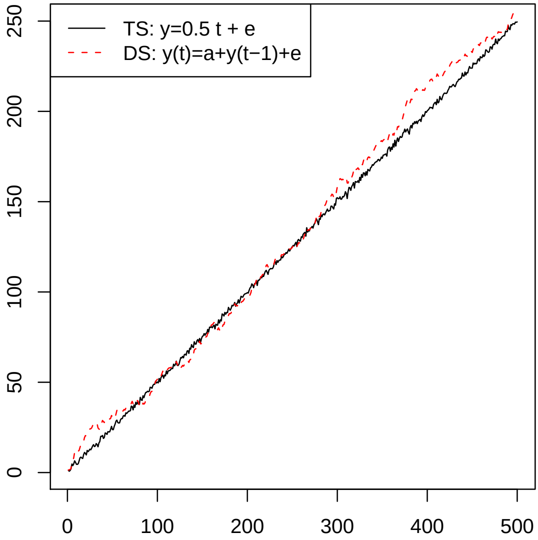
- $\mathrm{E}(X_t) = f(t)$
- $\mathrm{Var}(X_t) \neq f(t)$
- $\mathrm{Cov}(X_t, X_{t-j}) \neq f(t)$

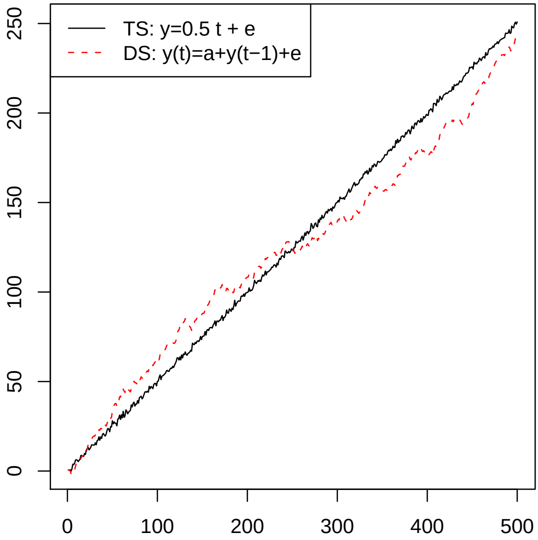
It is non-stationary as its expectation is time varying. However, its variance does not vary with time!

This process is called trend-stationary: if one detrends it, the series is stationary:

Proposition

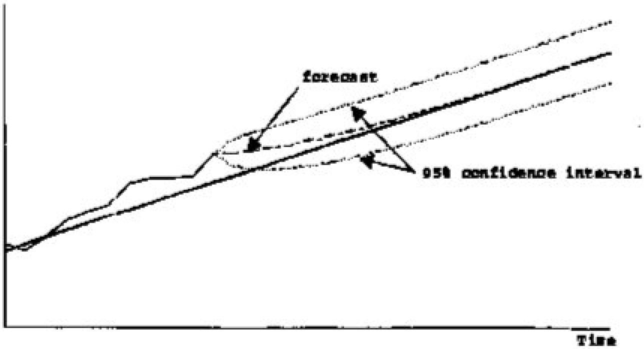
$Y_t \equiv a + bt + \varphi Y_{t-1} + \varepsilon_{t-1}$ *is not stationary*
$Y_t - bt = a + \varphi Y_{t-1} + \varepsilon_{t-1}$ *is stationary*


TS: y=0.5 t + e
DS: y(t)=a+y(t−1)+e
0
50
100
150
200
250
0
100
200
300
400
500

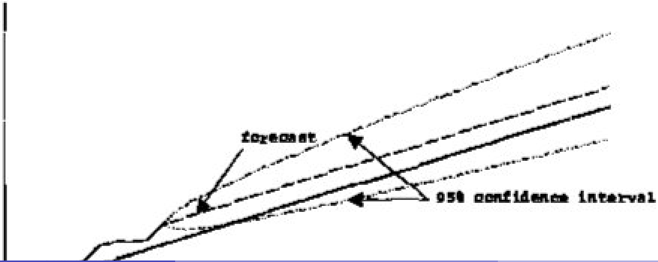

TS: y=0.5 t + e
DS: y(t)=a+y(t−1)+e
0
50
100
150
200
250
0
100
200
300
400
500

Comparisons between Trend and difference stationarity:

	TS	DS
DGP:	$Y_t = \alpha + \beta t + \varepsilon_t$	$Y_t = c + Y_{t-1} + \varepsilon_t$
DGP'		$Y_t = Y_0 + ct + \sum_{i=0}^{t-1} \varepsilon_{t-1}$
$\mathrm{E}(Y_t)$	$\alpha + \beta t$	$Y_0 + ct$
$\mathrm{Var}(Y_t)$	σ^2	$t\sigma^2$
$\mathrm{Cov}(Y_t, Y_{t-j})$	0	$(t-j)\sigma^2$
$\frac{\delta Y_t}{\delta \varepsilon_{t-i}}$	$\xrightarrow{\infty} 0$	$= a$
$\mathrm{E}[y_{t+s} - \hat{y}_{t+s\|t}]^2$	$\neq f(s)$	$= f(s)$



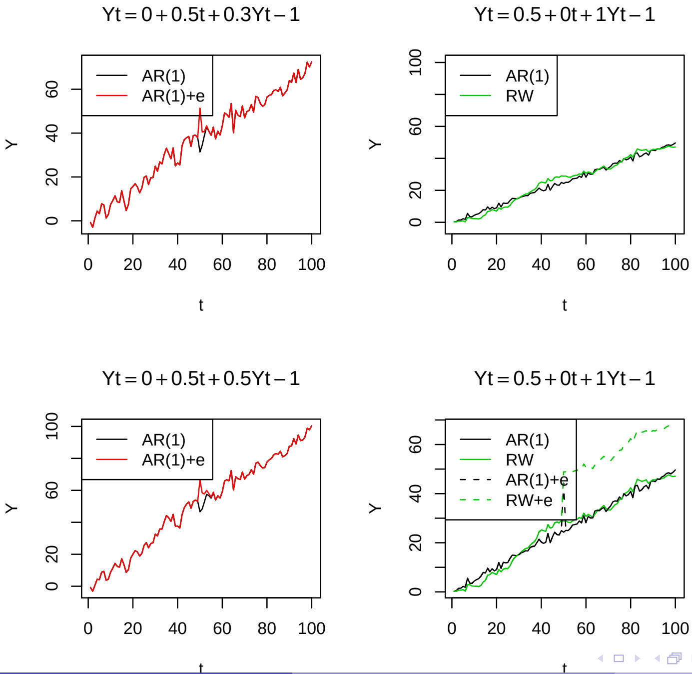
(a) Trend-stationary process



So both TS and DS exhibit a trend tendency but with stable or increasing variance.

This trend is said:

- Deterministic: TS process
- Stochastic: DS process


Yt = 0 + 0.5t + 0.3Yt − 1
AR(1)
AR(1)+e
Yt = 0.5 + 0t + 1Yt − 1
AR(1)
RW
Yt = 0 + 0.5t + 0.5Yt − 1
AR(1)
AR(1)+e
Yt = 0.5 + 0t + 1Yt − 1
AR(1)
RW
AR(1)+e
RW+e
Y
t

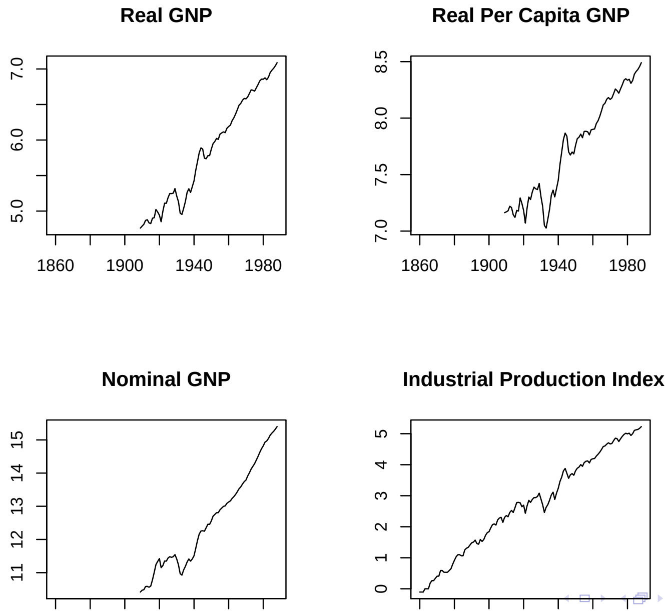
Outline

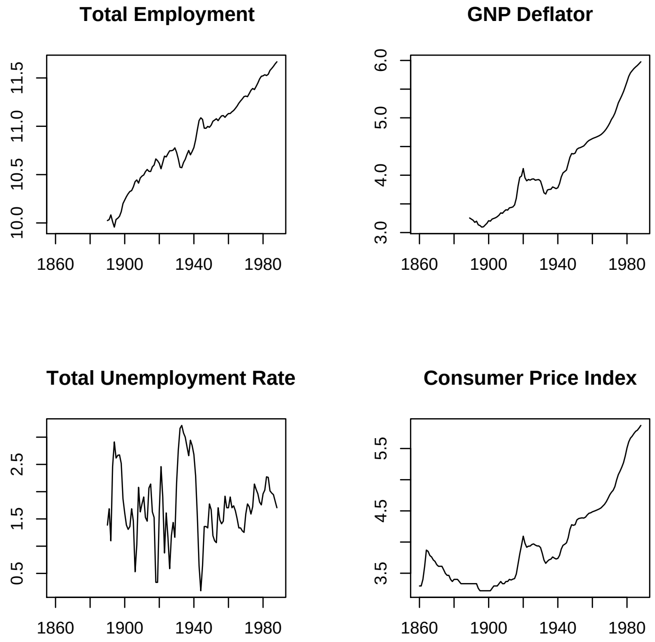
Nelson and Plosser (1982) study

Nelson and Plosser (1982) investigate 14 time series:

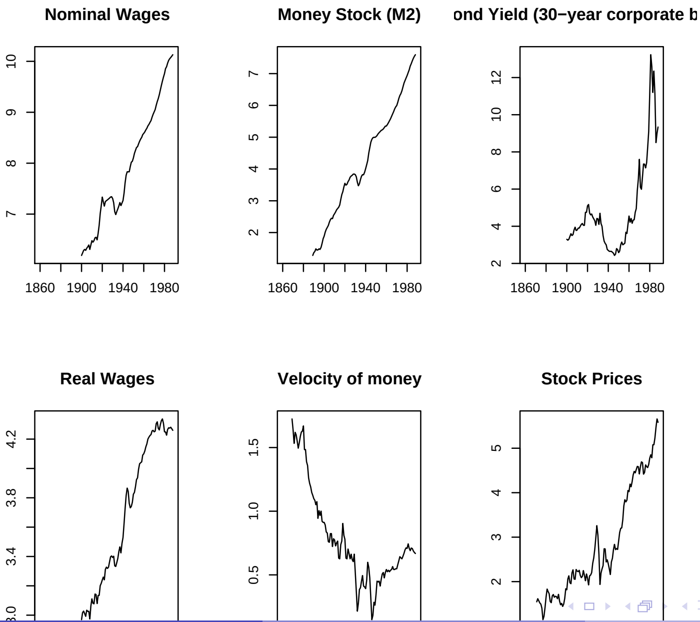
- Real GNP
- Nominal GNP
- Real Per Capita GNP
- Industrial Production Index
- Total Employment
- Total Unemployment Rate
- GNP Deflator
- Consumer Price Index
- Nominal Wages
- Real Wages
- Money Stock (M2)
- Velocity of money
- Bond Yield (30-year corporate bonds)
- Stock Prices

Nelson and Plosser (1982) study




Total Employment
10.0
10.5
11.0
11.5
1860
1900
1940
1980
GNP Deflator
3.0
4.0
5.0
6.0
1860
1900
1940
1980
Total Unemployment Rate
0.5
1.5
2.5
1860
1900
1940
1980
Consumer Price Index
3.5
4.5
5.5
1860
1900
1940
1980

Nelson and Plosser (1982) study 3



Results of the test of Trend stationary vs Difference stationary:

Results

13 series can be viewed as DS, one (unemployment) as TS.

> *The distinction between the two classes of processes is fundamental and aceptance of the purely stochastic view of non-stationarity has broad implications for our understanding of the nature of economic phenomena*

We conclude that macroeconomic models that focus on monetary disturbances as a source of purely transitory (stationary) fluctuations may never be successful in explaining a very large fraction of output fluctuations and that stochastic variation due to real factors is an essential element of any model of economic fluctuations.

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- Please check 35 and 36
- Give title plots 38 39
- Page 4: The axes should really begin with 0;0 so that only 2 lines of the rect should be visible
-