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   - Time series
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   - The random-walk
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   - Trend stationarity

6. Economic meaning and examples
Economies of scale and increasing returns

So-called Nobel Prize has been attributed to Krugman:

Economies of scale combined with reduced transport costs also help to explain why an increasingly larger share of the world population lives in cities and why similar economic activities are concentrated in the same locations. [This], in turn, stimulates further migration to cities.
The increasing returns are:

- Self production: Learning curve
- Common production: adoption/network externalities
How is a keyboard organised?

QWERTZ vs DVORAK costs

Number of people
Average cost
QWERTZ
DVORAK
QWERTZ vs DVORAK costs

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8. Structural VAR models
9. Cointegration the Engle and Granger approach
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6 Economic meaning and examples
What is a time serie?

In econometrics, we deal with three types of data:

- Cross individual data: \( X_i \)  \( i : \) individu
- Times series data: \( X_t \)  \( t : \) time
- Panel data: \( X_{it} \)  \( t : \) time

**Definition**

A time series is a variable \( X \) indexed by the time \( t \): \( X_t \quad t = 1, 2, \ldots, T \).

**Examples**

The time \( t \) can be annual, monthly, daily...

- Let be \( X_t \) the annual GDP.
- Let \( Y_t \) be the monthly temperature
- \( Z_t \) be the daily stocks
Some distinctions

The time series can:

- Take discrete or continuous values.
- Be measured at discrete (monthly, daily) or continuous time (signal processing, finance).
- Be measured at regular or irregular intervals.

In this lecture, usually we will refer to time series that take continuous values and are measured at discrete and regular intervals.
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Description of a time serie

To describe a time serie, we will concentrate on:

- Its Data Generating Process (DGP)
- The joint distribution of its elements
- Its “moments”:
  - Its expected value, $E[X_t] \equiv \mu_t$.
  - Its variance, $\text{Var}[X_t] \equiv \sigma^2 \equiv \gamma_0$.
  - Its autocovariance or autocorrelation of order $k$, $\text{Cov}[X_t, X_{t-k}] \equiv \gamma_k$. 

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Variance

The “second” moments play an important role in time series and have a slight different definition:

**Definition**

The **variance** of $X_t$, denoted by $\gamma_0(t)$, is given by:

$$\text{Var}(X_t) \equiv E \left[ (X_t - E(X_t))^2 \right]$$

**Definition**

The **autocovariance** of $X_t$ of order $k$, denoted by $\gamma_k(t)$, is given by:

$$\text{Cov}(X_t, X_{t-k}) \equiv E \left[ (X_t - E(X_t))(X_{t-k} - E(X_{t-k})) \right]$$

**Definition**

The **autocorrelation** of $X_t$ of order $k$, denoted by $\rho_k(t)$, is given by:

$$\text{Corr}(X_t, X_{t-k}) \equiv \frac{\text{Cov}(X_t, X_{t-k})}{\sqrt{\text{Var}(X_t)} \sqrt{\text{Var}(X_{t-k})}}$$
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The stationarity is an essential property to define a time series process:

**Definition**

A process is said to be **covariance-stationary**, or **weakly stationary**, if its first and second moments are **time invariant**.

\[
\begin{align*}
E(Y_t) &= E(Y_{t-1}) = \mu \quad \forall t \\
\text{Var}(Y_t) &= \gamma_0 < \infty \quad \forall t \\
\text{Cov}(Y_t, Y_{t-k}) &= \gamma_k \quad \forall t, \forall k
\end{align*}
\]
Weak stationarity

- The third condition states that the autocovariances only depend on the decay in the time but not in the time itself.
- Hence, the structure of the serie does not change with the time.
- If a process is stationary, $\gamma_k = \gamma_{-k}$
Mean reverting propriety

A stationary process has the propriety to be *mean reverting*:

- it will fluctuate around its mean.
- This mean will act as an attractor.
- It will cross the mean line an infinite number of ways.
Strong stationarity

**Definition**

\[ f(Z_1, Z_2, ..., Z_t) = f(Z_{1+k}, Z_{2+k}, ..., Z_{t+k}) \]

- Implies weak stationarity
- Definition not very useful as informations about the density function are difficult to obtain
ergodicity

Not absolutely useful...

**Definition**

\[
\lim_{n \to \infty} E \left[ | Y(y_i, \ldots, y_{i+k}) Z(y_{i+n}, \ldots, y_{i+n+l}) | \right] = \\
E [ Y(y_i, \ldots, y_{i+k}) ] \cdot E [ Z(y_{i+n}, \ldots, y_{i+n+l}) ]
\]
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Example 1

We define now the simplest stationary process:

**Definition**

An **independent white noise process** is a sequence of independent elements with same expected value and variance.

We will denote it $\varepsilon_t \sim iid(0, \sigma^2)$
Example 2

Consider the first order auto-regressive process AR(1) with a constant:

\[ Y_t = c + \varphi Y_{t-1} + \varepsilon_t \quad \varepsilon_t \sim iid(0, \sigma^2) \]

Theorem

An AR(1) process is asymptotically stationary if \( |\varphi| < 1 \)

We will need for its proof to remember the properties of a geometric progression:

Lemma (Infinite geometric progression)

\[ |\alpha| < 1 \iff \sum_{i=0}^{\infty} \alpha^i = 1 + \alpha + \alpha^2 + \alpha^3 + \ldots = \frac{1}{1-\alpha} \]
Proof.

The AR(1) can be written as:

\[ Y_t = c \sum_{i=0}^{t-1} \varphi^i + \varphi^t Y_0 + \sum_{i=0}^{t-1} \varphi^i \varepsilon_{t-i} \]

If \( |\varphi| < 1 \), it can be simplified into:

\[ Y_t = \frac{c}{1 - \varphi} + \sum_{i=0}^{t-1} \varphi^i \varepsilon_{t-i} \]

We have then:

- \( \mathbb{E}(X_t) = \frac{c}{1 - \varphi} \)
- \( \text{Var}(X_t) = \frac{\sigma^2}{1 - \varphi^2} \)
- \( \text{Cov}(X_t, X_{t-j}) = \frac{\varphi^j}{1 - \varphi^2} \sigma^2 \)
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Corollary

An AR(1) process is nonstationary if $|\varphi| \geq 1$. Furthermore, if:

- $|\varphi| = 1$ it is difference stationary
- $|\varphi| > 1$ it is explosive.
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Proof.

The AR(1) process without constant $Y_t = \varphi Y_{t-1} + \varepsilon_t$ with $|\varphi| = 1$ can be rewritten as:

$$Y_t = Y_0 + \sum_{i=0}^{t-1} \varepsilon_{t-i}$$

We have then:

- $E(X_t) = Y_0$
- $\text{Var}(X_t) = t\sigma^2$
- $\text{Cov}(X_t, X_{t-j}) = (t - j)\sigma^2$
The AR(1) process with $|\varphi| = 1$ is called a random walk. It is said to be difference stationary.

**Definition**

The difference operator takes the difference between a value of a time series and its lagged value. $\Delta X_t \equiv X_t - X_{t-1}$

**Definition**

A process is said to be difference stationary if it becomes stationary after being differenced once.

Note: a difference stationary process is also called integrated of order 1 and denoted by $X_t \sim I(1)$.
Theorem

A random walk is difference stationary.

Proof.

\[ \Delta Y_t = Y_t - Y_{t-1} = Y_{t-1} + \varepsilon_t - Y_{t-1} = \varepsilon_t \]
<table>
<thead>
<tr>
<th>Stationary AR(1)</th>
<th>Random Walk</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y_t = \varphi Y_{t-1} + \varepsilon_t$</td>
<td>$Y_t = Y_{t-1} + \varepsilon_t$</td>
</tr>
<tr>
<td>$</td>
<td>\varphi</td>
</tr>
<tr>
<td>$Y_t = Y_0 + \sum_{i=0}^{t-1} \varphi^i \varepsilon_{t-1}$</td>
<td>$Y_t = Y_0 + \sum_{i=0}^{t-1} \varepsilon_{t-1}$</td>
</tr>
<tr>
<td>$E(Y_t) = 0$ (if $Y_0 = 0$)</td>
<td>$E(Y_t) = 0$ (if $Y_0 = 0$)</td>
</tr>
<tr>
<td>$\text{Var}(Y_t) = \frac{\sigma^2}{1-\varphi^2}$</td>
<td>$\text{Var}(Y_t) = \sigma^2 t$</td>
</tr>
<tr>
<td>$\text{Cov}(Y_t, Y_{t-j}) = \frac{\varphi^j}{1-\varphi^2} \sigma^2$</td>
<td>$\text{Cov}(Y_t, Y_{t-j}) = (t - j) \sigma^2$</td>
</tr>
<tr>
<td>$\text{Corr}(Y_t, Y_{t-j}) = \varphi^j$</td>
<td>$\text{Corr}(Y_t, Y_{t-j}) = \sqrt{\frac{t-s}{t}} \xrightarrow{\infty} 1$</td>
</tr>
<tr>
<td>$\frac{\delta Y_t}{\delta \varepsilon_{t-i}} \xrightarrow{\infty} 0$</td>
<td>$\frac{\delta Y_t}{\delta \varepsilon_{t-i}} = 1$</td>
</tr>
<tr>
<td>Is mean reverting</td>
<td>Tends to move away from the mean</td>
</tr>
</tbody>
</table>
Simulation of AR(1) and RW

Random Walk and AR

\[ \text{Random Walk and AR (1)}: \rho = 0.4 \]

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Simulation of AR(1) and RW

Random Walk and AR

\[ \text{AR}(1) : \rho = 0.4 \]
Simulation of AR(1) and RW

Random Walk and AR

\[ \text{AR}(1) : \rho = 0.9 \]

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### Stationary AR(1)

\[ Y_t = \varphi Y_{t-1} + \varepsilon_t \]

- \(|\varphi| < 1\), \(\varepsilon_t \sim iid(0, \sigma^2)\)
- \(Y_t = Y_0 + \sum_{i=0}^{t-1} \varphi^i \varepsilon_{t-1}\)
- \(E(Y_t) = 0( \text{if } Y_0 = 0)\)
- \(\text{Var}(Y_t) = \frac{\sigma^2}{1-\varphi^2}\)
- \(\text{Cov}(Y_t, Y_{t-j}) = \frac{\varphi^j}{1-\varphi^2}\sigma^2\)
- \(\text{Corr}(Y_t, Y_{t-j}) = \varphi^j\)
- \(\frac{\delta Y_t}{\delta \varepsilon_{t-i}} \xrightarrow{\infty} 0\)

Is mean reverting

### Random Walk

\[ Y_t = Y_{t-1} + \varepsilon_t \]

- \(\varepsilon_t \sim iid(0, \sigma^2)\)
- \(Y_t = Y_0 + \sum_{i=0}^{t-1} \varepsilon_{t-1}\)
- \(E(Y_t) = 0( \text{if } Y_0 = 0)\)
- \(\text{Var}(Y_t) = \sigma^2 t\)
- \(\text{Cov}(Y_t, Y_{t-j}) = (t-j)\sigma^2\)
- \(\text{Corr}(Y_t, Y_{t-j}) = \sqrt{\frac{t-s}{t}} \xrightarrow{\infty} 1\)
- \(\frac{\delta Y_t}{\delta \varepsilon_{t-i}} = 1\)

Tends to move away from the mean
$Y_t = 0 + 0t + 0.3Y_{t-1}$

$Y_t = 0 + 0t + 0.5Y_{t-1}$

$Y_t = 0 + 0t + 0.95Y_{t-1}$

$Y_t = 0 + 0t + 1Y_{t-1}$
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The random walk with drift

\[ X_t = a + x_{t-1} + \varepsilon_t \quad \varepsilon_t \sim iid(0, \sigma^2) \]

Can be rewritten as:

\[ Y_t = at + Y_0 + \sum_{i=0}^{t-1} \varepsilon_{t-i} \]

Expectation is also time-varying:

- \( E(X_t) = Y_0 + at \)
- \( \text{Var}(X_t) = \ldots = f(t) \)
- \( \text{Cov}(X_t, X_{t-j}) = \ldots = f(t) \)

But it is still difference stationary:

Proof.

\[ \Delta Y_t = Y_t - Y_{t-1} = a + Y_{t-1} + \varepsilon_t - Y_{t-1} = a + \varepsilon_t \]
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Trend stationarity

\[ Y_t = a + bt + \varphi Y_{t-1} + \varepsilon_{t-1} \]

\[ Y_t = a \sum_{i=0}^{t-1} \varphi^i + a \sum_{i=0}^{t-1} \varphi^i + \varphi^t Y_0 + \sum_{i=0}^{t-1} \varphi^i \varepsilon_{t-i} \]

If \(|\varphi| < 1\), it can be simplified into:

\[ Y_t = \frac{a}{1-\varphi} + b \sum_{i=0}^{t-1} \varphi^i (t - i) + \sum_{i=0}^{t-1} \varphi^i \varepsilon_{t-i} \]
Proof.

\[ Y_t = a + bt + \varphi Y_{t-1} + \varepsilon_{t-1} \]

\[ = a + bt + \varphi(a + b(t - 1) + \varphi Y_{t-2} + \varepsilon_{t-2}) + \varepsilon_{t-1} \]

\[ = a(1 + \varphi) + bt + \varphi b(t - 1) + \varphi^2 Y_{t-2} + \varphi \varepsilon_{t-2} + \varepsilon_{t-1} \]

\[ = a(1 + \varphi) + bt + \varphi b(t - 1) + \varphi^2(a + b(t - 2) + \varphi Y_{t-3} + \varepsilon_{t-3}) \]

\[ + \varphi \varepsilon_{t-2} + \varepsilon_{t-1} \]

\[ = a(1 + \varphi + \varphi^2) + b(t + \varphi(t - 1) + \varphi^2(t - 2)) + \varphi^3 Y_{t-3} \]

\[ + \varphi \varepsilon_{t-3} + \varphi \varepsilon_{t-2} + \varepsilon_{t-1} \]

\[ = \ldots \]

\[ = \varphi^t Y_0 + a \sum_{i=0}^{t-1} \varphi^i + b \sum_{i=0}^{t-1} \varphi^i(t - i) + \sum_{i=0}^{t-1} \varphi^i \varepsilon_{t-i} \]

\[ = \frac{a}{1 - \varphi} + b \sum_{i=0}^{t-1} \varphi^i(t - i) + \sum_{i=0}^{t-1} \varphi^i \varepsilon_{t-i} \]
Thus the model \( Y_t = a + bt + \varphi Y_{t-1} + \varepsilon_{t-1} \) has:
- \( E(X_t) = f(t) \)
- \( \text{Var}(X_t) \neq f(t) \)
- \( \text{Cov}(X_t, X_{t-j}) \neq f(t) \)

It is non-stationary as its expectation is time varying. However, its variance does not vary with time!

This process is called trend-stationary: if one detrends it, the series is stationary:

**Proposition**

\[
Y_t \equiv a + bt + \varphi Y_{t-1} + \varepsilon_{t-1} \text{ is not stationary}
\]
\[
Y_t - bt = a + \varphi Y_{t-1} + \varepsilon_{t-1} \text{ is stationary}
\]
TS: $y = 0.5 \ t + e$

DS: $y(t) = a + y(t-1) + e$
TS: $y = 0.5 \, t + e$

DS: $y(t) = a + y(t-1) + e$
Comparisons between Trend and difference stationarity:

<table>
<thead>
<tr>
<th></th>
<th>TS</th>
<th>DS</th>
</tr>
</thead>
<tbody>
<tr>
<td>DGP:</td>
<td>$Y_t = \alpha + \beta t + \epsilon_t$</td>
<td>$Y_t = c + Y_{t-1} + \epsilon_t$</td>
</tr>
<tr>
<td>$DGP'$</td>
<td></td>
<td>$Y_t = Y_0 + ct + \sum_{i=0}^{t-1} \epsilon_{t-1}$</td>
</tr>
<tr>
<td>$E(Y_t)$</td>
<td>$\alpha + \beta t$</td>
<td>$Y_0 + ct$</td>
</tr>
<tr>
<td>$Var(Y_t)$</td>
<td>$\sigma^2$</td>
<td>$t\sigma^2$</td>
</tr>
<tr>
<td>$Cov(Y_t, Y_{t-j})$</td>
<td>0</td>
<td>$(t-j)\sigma^2$</td>
</tr>
<tr>
<td>$\frac{\delta Y_t}{\delta \epsilon_{t-i}}$</td>
<td>$\infty \rightarrow 0$</td>
<td>$= a$</td>
</tr>
<tr>
<td>$E[y_{t+s} - \hat{y}_{t+s}</td>
<td>t]^2$</td>
<td>$\neq f(s)$</td>
</tr>
</tbody>
</table>
(a) Trend-stationary process
So both TS and DS exhibit a trend tendency but with stable or increasing variance.

This trend is said:

- Deterministic: TS process
- Stochastic: DS process
$Y_t = 0 + 0.5t + 0.3Y_{t-1}$

$Y_t = 0.5 + 0t + 1Y_{t-1}$

$Y_t = 0 + 0.5t + 0.5Y_{t-1}$

$Y_t = 0.5 + 0t + 1Y_{t-1}$
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Nelson and Plosser (1982) study

Nelson and Plosser (1982) investigate 14 time series:

- Real GNP
- Nominal GNP
- Real Per Capita GNP
- Industrial Production Index
- Total Employment
- Total Unemployment Rate
- GNP Deflator
- Consumer Price Index
- Nominal Wages
- Real Wages
- Money Stock (M2)
- Velocity of money
- Bond Yield (30-year corporate bonds)
- Stock Prices
Nelson and Plosser (1982) study

Real GNP

Real Per Capita GNP

Nominal GNP

Industrial Production Index
Nelson and Plosser (1982) study 3
Results of the test of Trend stationary vs Difference stationary:

Results

13 series can be viewed as DS, one (unemployment) as TS.

The distinction between the two classes of processes is fundamental and acceptance of the purely stochastic view of non-stationarity has broad implications for our understanding of the nature of economic phenomena.
We conclude that macroeconomic models that focus on monetary disturbances as a source of purely transitory (stationary) fluctuations may never be successful in explaining a very large fraction of output fluctuations and that stochastic variation due to real factors is an essential element of any model of economic fluctuations.
- Fill var and cov page 33
- Please check 35 and 36
- Give title plots 38 39
- Page 4: The axes should really begin with 0;0 so that only 2 lines of the rect should be visible