Structural VAR models

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December 9, 2008

Lectures list

1. Stationarity
2. ARMA models for stationary variables
3. Some extensions of the ARMA model
4. Non-stationarity
5. Seasonality
6. Non-linearities
7. Multivariate models
8. **Structural VAR models**
9. Cointegration the Engle and Granger approach
10. Cointegration 2: The Johansen Methodology
11. Multivariate Nonlinearities in VAR models
12. Multivariate Nonlinearities in VECM models

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VMA representation

VAR(1) to $VMA(\infty)$:

$$y_t = A_1 y_{t-1} + \varepsilon_t \tag{1}$$
$$= A_0 + \sum_{i=0}^{\infty} A_1^i \varepsilon_{t-i} \tag{2}$$

VAR(p) in VAR(1) to $VMA(\infty)$:

$$Y_t = \mathbf{A}_1 Y_{t-1} + E_t \tag{3}$$
$$= \sum_{i=0}^{\infty} \mathbf{A}_1^i E_{t-i} \tag{4}$$

Interpretation of the $VMA(\infty)$ coefficient matrices

From the $VMA(\infty)$ of a VAR(1):

$$y_t = A_0 + \sum_{i=0}^{\infty} A_1^i \varepsilon_{t-i}$$

Elements of A^i represents effects of unit shocks in the variables after i periods.
Interpretation: the ε_t can be seen as 1 step-ahead forecast error so are called *forecasr error impulse responses*.

```
>   library(vars)
> data(Canada)
> var<- VAR(Canada[,c("e","rw")], p = 2, type = "const")
> imp<-irf(var, boot=FALSE, ortho=FALSE,n.ahead=200)
> plot(imp)
```

Scaling of the impulse:
rescale the axes with $1 = \sqrt{\sigma_y^2}$

Relation with Granger causality:
IRF of y_1 to $y_i \quad i \neq 1$ is zero if y_1 does not Granger cause the others variables

Cumulated IRF

From MA representation:

$$y_t = A_0 + \sum_{i=0}^{\infty} \Phi_i \varepsilon_{t-i}$$

We now want to know the cumulated impact:
$\Psi_n = \sum_{i=0}^{n} \Phi_i$ accumulated responses over n periods

problem with the interpretation

Previous assumption: schocks are independent (we force other shocks to be zero).

But shocks may be correlated! See the var-cov matrix of the residuals.

So impulse is composite effect and interpretation is not direct.

Solution: Create independant (orthogonal) residuals.

Triangular and Choleski decomposition of matrices

Theorem (Triangular decomposition)

Any positive definite symmetric matrix A has a unique representation of the form:

$$A = BDB^{'}$$

where:

- *B is lower triangle with 1 along the principal diagonal*
- *D is a diagonal matrix*

Theorem (Choleski decomposition)

Any positive definite symmetric matrix A has a unique representation:

$$A = PP^{'}$$

P is lower triangle with squares roots of D along the diagonal

Choleski decomposition just let $P = BD^{1/2}$

Choleski decomposition with R

```
>  m <- matrix(c(5,1,1,3),2,2);m

     [,1] [,2]
[1,]    5    1
[2,]    1    3

> tP <- chol(m);tP

         [,1]      [,2]
[1,] 2.236068 0.4472136
[2,] 0.000000 1.6733201

> #R gives P'P (and we saw PP')
> t(tP) %*% tP

     [,1] [,2]
[1,]    5    1
[2,]    1    3
```

Triangular decomposition with R

It is not available directly so we find it from Choleski by setting:

- C has the diag of P
- $B = PC^{-1}$
- $D = CC'$

```
> C<-matrix(0,2,2)
> diag(C)<-diag(tP)
> A<-t(tP)%*%solve(C);A #lower triangle
     [,1] [,2]
[1,]  1.0    0
[2,]  0.2    1
> D<-C%*%t(C);D #diagonal
     [,1] [,2]
[1,]    5  0.0
[2,]    0  2.8
> A%*%D%*%t(A) #original matrix
     [,1] [,2]
[1,]    5    1
[2,]    1    3
```

Orthogonal IRF

From the VMA:

$$y_t = A_0 + \sum_{i=0}^{\infty} \Phi \varepsilon_{t-i}$$

Decompose $\Sigma_\varepsilon = PP'$ where P is lower triangular matrix.
Insert PP^{-1} into the VMA:

$$y_t = A_0 + \sum_{i=0}^{\infty} \Phi_i PP^{-1} \varepsilon_{t-i}$$

And let:

- $\Theta_i \equiv \Phi_i P$
- $w_t \equiv P^{-1} \varepsilon_t$

So we have:

$$y_t = A_0 + \sum_{i=0}^{\infty} \Theta w_{t-i}$$

Orthogonal IRF 2

We rewrote

$$y_t = A_0 + \sum_{i=0}^{\infty} \Theta w_{t-i}$$

Proposition

The residuals w_t are independant to each other: $\text{Var}(w_t) = I$

Proof.

$\text{Var}(w_t) \equiv \Sigma_w = \text{Var}(P^{-1}\varepsilon_t) = P^{-1}\Sigma_\varepsilon P^{-1'} = P^{-1}PP'P^{-1'} = I$ □

Hence:

- Innovations are independant
- Variance is one

So unit innovation is just an innovation of size one standard deviation.

The (j,k) element of Θ_j is assumed to represent the effect on variable j of a unit innovation of variable k thats has occured i period ago.

Orthogonal IRF 3

This orthogonalization is called sometimes Wold

Outline

SVAR model

We now study a full system:

$$\begin{aligned} y_t &= \delta_1 + b_{12} z_t + \gamma_{11} y_{t-1} + \gamma_{12} z_{t-1} + \varepsilon_{1t} \\ z_t &= \delta_2 + b_{21} y_t + \gamma_{21} y_{t-1} + \gamma_{22} z_{t-1} + \varepsilon_{2t} \end{aligned}$$

In matrix notation:

$$\begin{bmatrix} 1 & -b_{12} \\ -b_{21} & 1 \end{bmatrix} \begin{bmatrix} y_t \\ z_t \end{bmatrix} = \begin{bmatrix} \delta 1 \\ \delta_2 \end{bmatrix} + \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}$$

Compactly:

$$Bx_t = \Delta + \Gamma x_{t-1} + \varepsilon_t$$

From SVAR to VAR

This SVAR can be rewritten in a VAR by premultiplying by B^{-1}:

$$x_t = A_0 + A_1 x_{t-1} + u_t$$

- $u_t = B^{-1}\varepsilon_t$
- $A_0 = B^{-1}\Delta$
- $A_1 = B^{-1}\Gamma$

Under the assumption that the ε_t from SVAR are white noise, the u_t from VAR have:

- Zero mean
- fixed variance
- individually not autocorelated
- Correlated to each other if $b_{12}, b_{21} \neq 0$

Examine: $u_t = B^{-1}\varepsilon_t = \begin{cases} u_{1t} & = (\varepsilon_{1t} - b_{12}\varepsilon_{2t})/(1 - b_{12}b_{21}) \\ u_{2t} & = (\varepsilon_{2t} - b_{21}\varepsilon_{1t})/(1 - b_{12}b_{21}) \end{cases}$

Problems with the estimation

- SVAR: endogeneity problem
- VAR to SVAR: unidentification of the structural parameters. Need restrictions

Example

- VAR: 9=2+4+3 parameters (intercept+slope+var/cov)
- SVAR: 10=2+2+4+2 parameters (contemp+intercept+slope+var)

We have to make 1 restriction on the structural parameters

Proposition

We need $\frac{k^2-k}{2}$ restrictions to identify the SVAR from the VAR.

Types of restrictions

- Choleski
- Triangular
- Economic a priori
- Blanchard-Quah

Choleski restrictions

Remember, by setting:

$$w_t \equiv P^{-1}\varepsilon_t \qquad \text{where } \Sigma_\varepsilon = PP'$$

we could go from a VMA with correlated residuals to a VMA with $\Sigma_w = I$:

$$y_t = A_0 + \sum_{i=0}^{\infty} \Theta w_{t-i}$$

This is implicitly a SVAR where $B = P$:

SVAR: $Px_t = \Delta + \Gamma x_{t-1} + \varepsilon_t$
VAR: $x_t = P^{-1}\Delta + P^{-1}\Gamma x_{t-1} + P^{-1}\varepsilon_t = A^* + A^* x_{t-1} + u_t$
The residuals are $\Sigma_u = \text{Var}(P^{-1}\varepsilon) = P^{-1}\Sigma_\varepsilon P^{-1'} = I$

Triangular restrictions

We can also use the triangular decomposition:

$$\Sigma_\varepsilon = ADA^{'}$$

and set $B = A$ so:

<u>SVAR:</u> $Ax_t = \Delta + \Gamma x_{t-1} + \varepsilon_t$
<u>VAR:</u> $x_t = A^{-1}\Delta + A^{-1}\Gamma x_{t-1} + A^{-1}\varepsilon_t = A^* + A^* x_{t-1} + u_t$
The residuals are $\Sigma_u = \text{Var}(A\varepsilon) = A\Sigma_\varepsilon A^{'} = D$

IRF

Both triangular or Choleski imply that B is lower triangular:
$B = \begin{pmatrix} 1 & 0 \\ b_{21} & 1 \end{pmatrix}$
That means that in the SVAR we have set $b_{12} = 0$:

$$y_t = \delta_1 + \overbrace{b_{12}}^{=0} z_t + \gamma_{11} y_{t-1} + \gamma_{12} z_{t-1} + \varepsilon_{1t}$$
$$z_t = \delta_2 + b_{21} y_t + \gamma_{21} y_{t-1} + \gamma_{22} z_{t-1} + \varepsilon_{2t}$$

This is a kind of **exogeneity**: we assume that there is **no contemporaneous impact** (or *Wold causality*)of z_t to y_t

This operation is called *ordering of the variables* and is made with economic arguments.

IRF

With the Choleski decomposition we can now compute the Impulse Response Function and interpret it as the impact of shocks of x_i to x_j.

Problem

The results differ when the ordering is changed!

Choleski

Remember the relation between the SVAR residuals u_t and the VAR residuals ε_t:

$$\varepsilon_t = Bu_t$$

$$u_{1t} = (\varepsilon_{1t} - \overbrace{b_{12}}^{=0} \varepsilon_{2t})/(1 - \overbrace{b_{12}}^{=0} b_{21}) = \qquad \varepsilon_{1t}$$

$$u_{2t} = (\varepsilon_{2t} - b_{21}\varepsilon_{1t})(1 - \overbrace{b_{12}}^{=0} b_{21}) = \varepsilon_{2t} - b_{21}\varepsilon_{1t}$$

Other restrictions:

- Coefficient: $b_{12} = 0$
- Variance: $\text{Var}(\varepsilon_{1t}) = 1$
- Symmetry: $b_{12} = b_{21}$

Cho decomposition

How do we obtain the B?

- $\widehat{\text{Var}}(u_t) = \hat{\Omega}$ from reduced VAR (estimated)
- Define $\text{Var}(u\varepsilon_t) \equiv \Sigma$ in the SVAR (not observed)
- Theoretical relationship: $u_t = B^{-1}\varepsilon_t$

So we obtain: $\hat{\Omega} = B^{-1}\Sigma B^{-1'}$
If we make further the assumption that $\Sigma = I$ we have:
$\hat{\Omega} = B^{-1}B^{-1'}$
Under the assumption that $P = B^{-1}$ is lower triangular matrix, we have the unique decomposition (Choleski): $\Omega = PP^{'}$

SVAR

$$B_0 y_t = c_0 + B_1 y_{t-1} + B_2 y_{t-2} + \cdots + B_p y_{t-p} + \epsilon_t$$

$$\begin{bmatrix} 1 & B_{0;1,2} \\ B_{0;2,1} & 1 \end{bmatrix} \begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \begin{bmatrix} c_{0;1} \\ c_{0;2} \end{bmatrix} + \begin{bmatrix} B_{1;1,1} & B_{1;1,2} \\ B_{1;2,1} & B_{1;2,2} \end{bmatrix} \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

Problem: results depend on the ordering of the variables! See sensitivity analysis during conference: tested for many different variables and obtain same result. Here same variables but different output.

Outline

1 Lectures

2 Impulse response function

3 Structural VAR models
- Stuctural vector autoregressive model (SVAR)
  - Choleski decomposition
  - Blanchard-Quah decomposition
- Impulse-response function
- Forecast variance decomposition

4 Impulse response function
- Stable VAR case

Outline

Forecast of a VAR(1)

Take the VAR(1):

Outline